A Cubic Facet Based Method for Estimating the Principal Quadric

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Abstract: A principal quadric estimator is described which is based on independently fitting cubic facets to the 3 components of position in a small neighbourhood of each data point. From the parameters of the fitted facets, the first and second order differential properties of the surface at the data point, and hence the principal quadric parameters, can be estimated. This method is compared with several other methods for estimating the principal quadric and is found to much more robust with respect to noise.

Keywords: Principal quadric, facets, estimation.

1 Introduction

The principal quadric characterises the local surface shape properties on a smooth (2D) surface in 3D space. Consequently, it is often estimated from range data obtained off such a surface as a first step in processing that data. This paper considers a method for estimating the principal quadric which is based on fitting cubic facets independently to the three components of position. The second order differential properties of the surface can then be estimated from the parameters of the facets, and the principal quadric calculated from them. This method is compared with other common methods such as directly fitting a quadratic surface patch to the data. It is shown to be more computationally efficient and generate more reliable estimates when the data contains noise.

The principal quadric is useful in range data analysis because it provides information about the local surface shape. The surface normal forms one of the axes of the principal frame (the coordinate frame that the principal quadric is defined in). The surface normal is often used as a cue in matching surface patches, detecting surface folds, etc. The Gaussian and mean curvatures can be used to parameterise the principal quadric. These are commonly used to classify surface points into 8 generic classes (planar surfaces, concave/convex parabolic surfaces, concave/convex elliptic surfaces, and concave/convex/symmetric hyperbolic surfaces), such as in [3]. An alternative use is the classification of surface patches on the basis of shape index and curvedness [2, 6]. Principal quadric data is also used in the segmentation of range image data [1, 8].

The rest of the paper is organised as follows. In Section 2, the estimator is described in detail. In Section 3, its performance is compared with that of other estimators. Finally, there is a conclusion in Section 4.

2 Description of Estimator

The principal quadric is a second order local descriptor of a smooth surface. It consists of a coordinate frame (the principal frame) formed from the surface normal and the two directions of principal curvature. In this coordinate frame, the principal quadric is parameterised by only two parameters.

Consider a range image given by a set of points in 3D space indexed by coordinates on a regular grid, that is

$$\mathcal{I} = \{\mathbf{X}(u, v) \in R^3 | u \in [0 \dots M-1], v \in [0 \dots N-1]\} \tag{1}$$

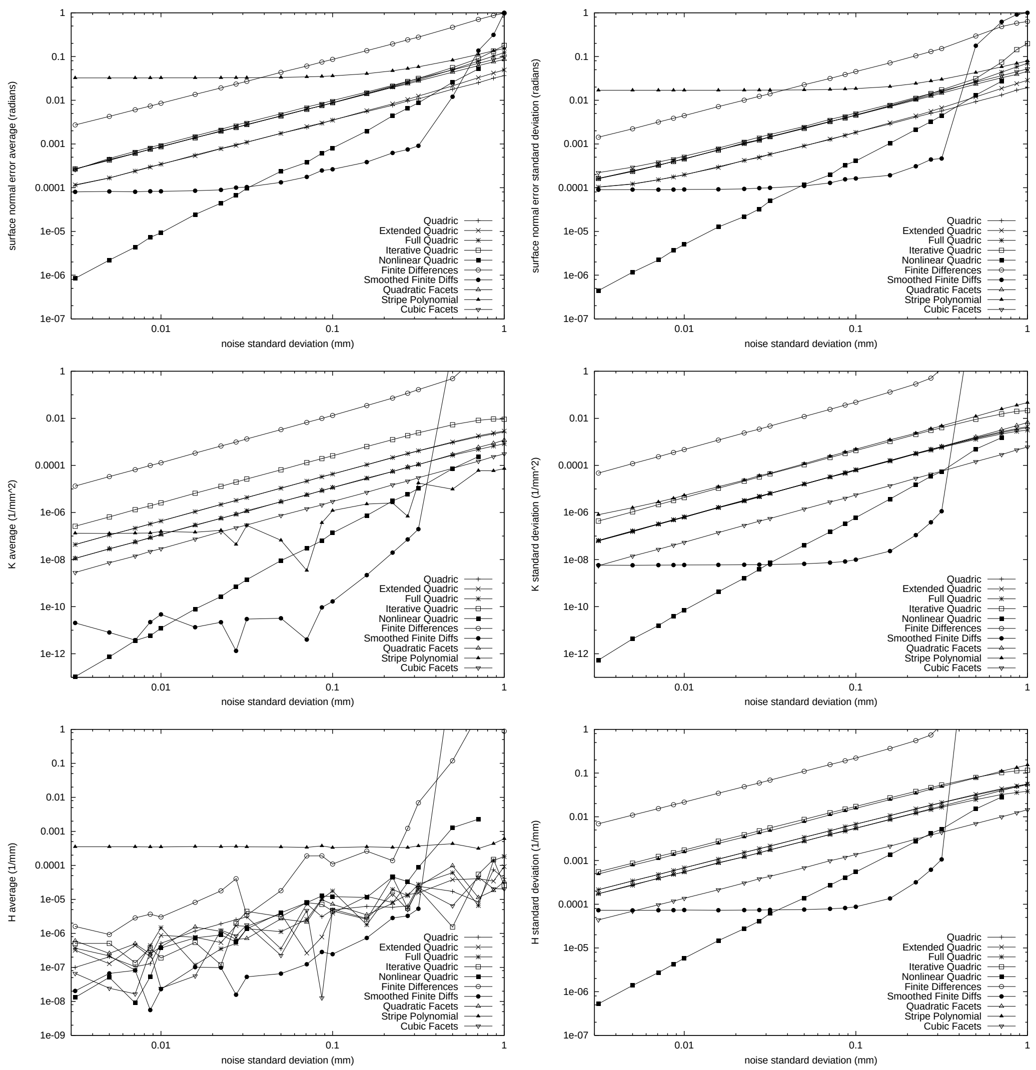

Figure 1: Performance statistics for a planar surface.

For example, a temporally encoded structured light system generates a 3D point for each pixel on its camera image plane. At every (u, v), a cubic facet [5, Vol 1, pg. 371] is fitted to each of the three components of $\mathbf{X}(u, v)$, to give the approximation

$$\begin{aligned} \mathbf{X}(u+i, v+j) &\approx \mathbf{F}(i,j) \\ &= \mathbf{f}_0 i^3 + \mathbf{f}_1 i^2 j + \mathbf{f}_2 i j^2 + \mathbf{f}_3 j^3 + \mathbf{f}_4 i^2 + \mathbf{f}_5 i j + \mathbf{f}_6 j^2 + \mathbf{f}_7 i + \mathbf{f}_8 j + \mathbf{f}_9 \end{aligned} \tag{2}$$

where the dependence of $\mathbf{F}$ and the $\mathbf{f}_l$ on (u, v) is not shown. The facet fitting is done over some neighbourhood of (u, v) in the range image. In the tested implementation, rectangular neighbourhoods centred on (u, v) were used. The parameters $\mathbf{f}_l$ were estimated using the linear least squares procedure to minimise the error

$$E = \sum_{(i,j)\in\mathcal{N}} (\mathbf{X}(u+i, v+j) - \mathbf{F}(i,j))^{\mathsf{T}} (\mathbf{X}(u+i, v+j) - \mathbf{F}(i,j)) \tag{3}$$

where $\mathcal{N}$ is the neighbourhood of (u, v).

Given the facet parameters, the first and second order derivatives of $\mathbf{F}(i,j)$ with respect to i and j can be calculated and used as estimates of the derivatives of $\mathbf{X}(u,v)$ at (u,v):

$$\begin{aligned}
\frac{\partial \mathbf{X}}{\partial u} &= \mathbf{f}_7 &\quad (4)\\
\frac{\partial \mathbf{X}}{\partial v} &= \mathbf{f}_8 &\quad (5)\\
\frac{\partial \mathbf{X}^2}{\partial u^2} &= 2\mathbf{f}_4 &\quad (6)\\
\frac{\partial \mathbf{X}^2}{\partial u \partial v} &= \mathbf{f}_5 &\quad (7)\\
\frac{\partial \mathbf{X}^2}{\partial v^2} &= 2\mathbf{f}_6 &\quad (8)
\end{aligned}$$

From these the parameters of the principal quadric (i.e., directions of the axes of the principal frame, and the Gaussian and mean curvatures) can be determined directly. The surface normal is given by

$$\mathbf{n} = \pm \frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} / \| \frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} \| \tag{9}$$

and the two principal curvatures by the solutions to

$$|\mathbf{G}|\kappa_i^2 - \alpha\kappa_i + |\mathbf{D}| = 0 \tag{10}$$

where

$$\begin{aligned}
\alpha &= g_{11}d_{22} + g_{22}d_{11} - 2g_{12}d_{12} &\quad (11)\\
\mathbf{G} &= \begin{bmatrix} \frac{\partial \mathbf{X}}{\partial u} \cdot \frac{\partial \mathbf{X}}{\partial u} & \frac{\partial \mathbf{X}}{\partial u} \cdot \frac{\partial \mathbf{X}}{\partial v} \\ \frac{\partial \mathbf{X}}{\partial v} \cdot \frac{\partial \mathbf{X}}{\partial u} & \frac{\partial \mathbf{X}}{\partial v} \cdot \frac{\partial \mathbf{X}}{\partial v} \end{bmatrix} &\quad (12)\\
\mathbf{D} &= \begin{bmatrix} \mathbf{n} \cdot \frac{\partial^2 \mathbf{X}}{\partial u^2} & \mathbf{n} \cdot \frac{\partial^2 \mathbf{X}}{\partial u \partial v} \\ \mathbf{n} \cdot \frac{\partial^2 \mathbf{X}}{\partial v \partial u} & \mathbf{n} \cdot \frac{\partial^2 \mathbf{X}}{\partial v^2} \end{bmatrix} &\quad (13)
\end{aligned}$$

and the principal directions of curvature are

$$\dot{\mathbf{X}}_i = \mathbf{A}\dot{\mathbf{u}}_i \tag{14}$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{\partial \mathbf{X}}{\partial u} & \frac{\partial \mathbf{X}}{\partial v} \end{bmatrix} \tag{15}$$

and $\dot{\mathbf{u}}_i$ is the solution to

$$(\mathbf{D} - \kappa_i \mathbf{G})\dot{\mathbf{u}}_i = 0 \tag{16}$$

for each of the principal curvatures κ_i.

The surface normal $\mathbf{n}$ forms the Z axis of the principal frame, and the other two are formed by the directions of principal curvature $\dot{\mathbf{X}}_i$. In this coordinate frame, the principal quadric is

$$Z = \frac{1}{2}(\kappa_1 X^2 + \kappa_2 Y^2) \tag{17}$$

Note that a smoothed estimate of the data point is also given by the facets:

$$\mathbf{X}\widehat{(u,v)} = \mathbf{f}_9 \tag{18}$$

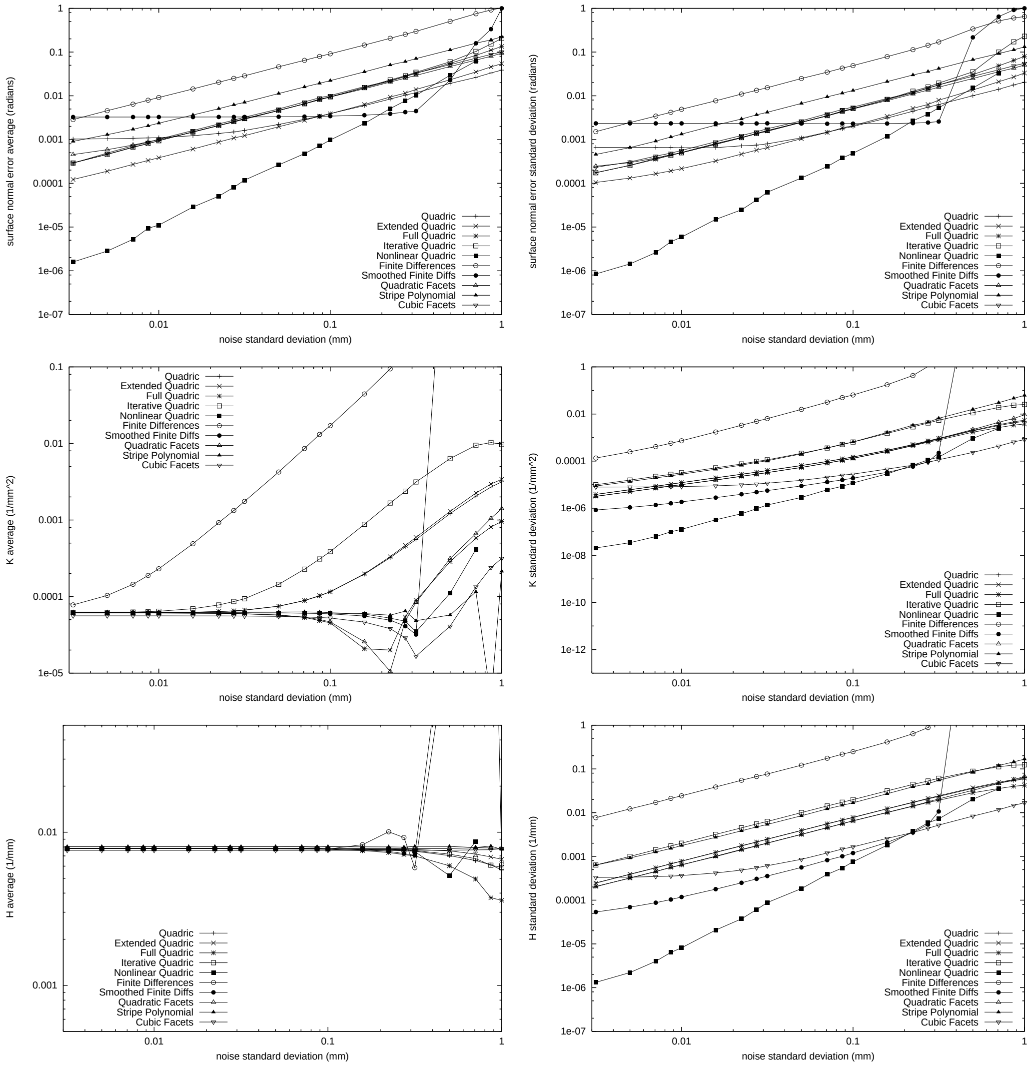

Figure 2: Performance statistics for a spherical surface.

3 Comparison With Other Methods

The cubic facet based principal quadric estimator was compared with other operators using the test procedure reported in [7]. These tests use a Monte Carlo technique based on synthetic data to measure the performance of the estimator on a selection of simple shapes (planes, spheres, cylinders) as a function of added Gaussian noise. This is repeated for a range of noise standard deviation values. Each principal quadric estimator was then applied to the data, and the errors (i.e. difference from the true value) in the surface normal, Gaussian curvature, and mean curvature calculated. Sample mean and standard deviations of these errors were then obtained from many trials. These are shown in Figures 1–3.

The other 9 methods tested are:

- quadric — estimate surface normal by total least squares planar fit, map data so that Z is in this direction, fit $Z = aX^2 + bXY + cY^2$.
- extended quadric — same as above but fit $Z = aX^2 + bXY + cY^2 + dX + eY$.
- full quadric — same as above but fit $Z = aX^2 + bXY + cY^2 + dX + eY + f$.

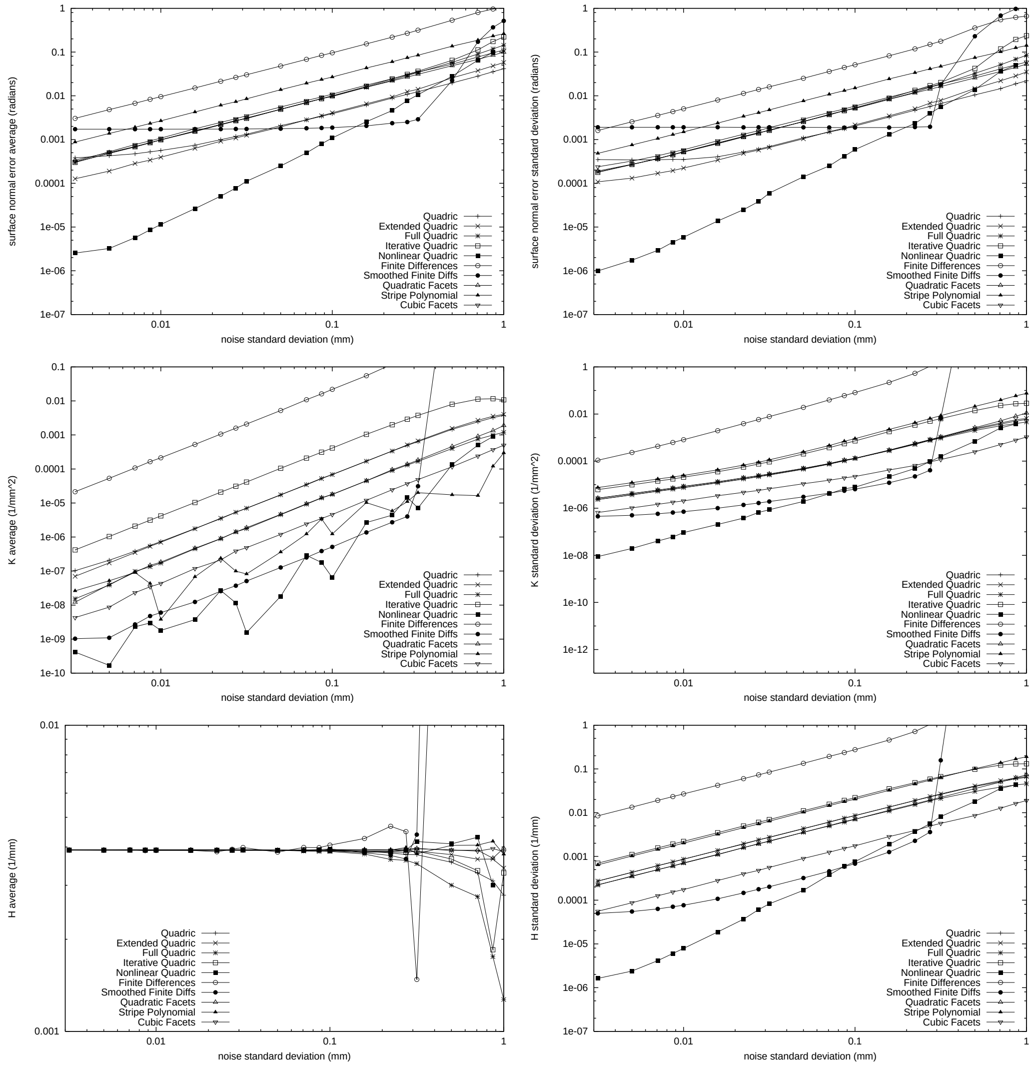

Figure 3: Performance statistics for a cylindrical surface.

- nonlinear quadric — fit a quadric directly by nonlinear least squares.
- finite differences — simple 3×3 derivative estimators.
- smoothed finite differences — same as above but with Gaussian pre-smoothing.
- quadratic facets — as for cubic facets but only second order terms in facet.
- stripe polynomial — fitting cubic to raw temporally encoded structured light system stripe data, a method specific to this type of sensor.

Looking at these results, the only methods that exhibit any significant bias in the estimates are the finite difference and stripe polynomial methods. The nonlinear quadric method has the smallest standard deviation but is the most computationally expensive. The smoothed finite difference method also has low standard deviation but the smoothing introduces significant artifacts on real images. The cubic facet method is generally the next best operator.

The cubic facet based method is a significant improvement over a similar quadratic facet based method, for a very small increment in computational complexity. It is also significantly better (in terms of having smaller variance of the parameter estimates) than directly fitting a quadratic surface patch. This is partly explained by the higher number of degrees of freedom in the cubic facet based method. The cubic facet based method has significantly less computational complexity than the direct surface fitting approaches. This is because the neighbourhood basis (i.e., the values of (u, v) in the basis) is the same at all points. Thus the design matrices are the same and their singular value decomposition only needs to be performed once per image instead of once per data point.

In this experimental evaluation, the (u, v) parameters are modelled as those of a camera image plane, and thus are related to the $\mathbf{X}(u, v)$ parameters by a pin-hole camera model projection. In [4], curvature estimation is evaluated under the assumption that the (u, v) parameters are formed from the orthogonal projection of $\mathbf{X}(u, v)$ onto the X–Y plane, and that all errors are perpendicular to this plane. In this scenario, similar conclusions are reached about the performance of facet fitting methods.

4 Conclusion

In this paper, a method has been presented for estimating the principal quadric by fitting cubic facets to the components of 3D position of the data points in a neighbourhood. From the cubic facet parameters, the first and second order derivatives can be estimated. The principal quadric can then be determined from these. This method was then compared with several other methods for estimating the principal quadric. The cubic facet method was shown to be superior to the majority of the other estimators. Only two methods have smaller standard deviation than the cubic facet based method: fitting a quadratic patch to the data by non-linear optimisation; and, using finite difference estimates after the data has been spatially smoothed with a large Gaussian weighted filter. The non-linear optimisation based method has a significantly larger computational cost. The finite difference with pre-smoothing based estimator has a significant bias in its estimates as the smoothing effectively flattens the underlying surface.

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